Open and Short induction in $K(F_{\mathsf{rud}}, G_{\mathsf{rud}})$

Open induction

We fix an open $L_n(F_{\mathsf{rud}}, G_{\mathsf{rud}})$ formula with a single free variable $A(x)$.
The goal for this subsection is to prove that, for arbitrary $\alpha \in F_{\mathsf{rud}}$

$$[\![\neg A(0) \vee A(\alpha) \vee \exists x < \alpha(A(x) \wedge \neg A(x+1))]\!] = 1_{\mathcal{B}}, \tag{1}$$

where $0 \in F_{\mathsf{rud}}$ is the constant 0 function.

The above equation is, of course, equivalent to

$$([\![A(0)]\!] \wedge [\![\neg A(\alpha)]\!]) \leq ([\![\beta < \alpha]\!] \wedge [\![A(\beta)]\!] \wedge [\![\neg A(\beta+1)]\!]) \tag{2}$$

for a suitable $\beta \in F_{\mathsf{rud}}$.

Exercise 1. Show that we can w.l.o.g. assume that both $\langle\!\langle A(0)\rangle\!\rangle$ and $\langle\!\langle \neg A(\alpha)\rangle\!\rangle$ equal Ω. In particular, this implies $[\![A(0)]\!] \wedge [\![\neg A(\alpha)]\!] = 1_{\mathcal{B}}$.

Exercise 2. (*optional*) While we will not use it, we may as well assume that α is a constant function mapping each sample ω to a number $m \in \mathbb{M}_n$. Argue that this is, indeed, the case.

Exercise 3. Find a suitable β satisfying $\langle\!\langle \beta < \alpha \wedge A(\beta) \wedge \neg A(\beta+1)\rangle\!\rangle = \Omega$ (the binary search might come in handy).

Short induction

The previous subsection established that the usual induction (for open formulas) is valid in $K(F_{\mathsf{rud}}, G_{\mathsf{rud}})$. This is equivalent to the statement that the induction (for open formulas) holds true up to an arbitrary $\alpha \in F_{\mathsf{rud}}$, or even up to an arbitrary $m \in \mathbb{M}_n$ taken as the corresponding constant function.

The **short induction** then refers to the statement that the usual induction axiom holds true up to $\log(m)$ for arbitrary $m \in \mathbb{M}_n$.

Exercise 4. Assuming you were to prove that $K(F_{\mathsf{rud}}, G_{\mathsf{rud}})$ satisfies short induction for open formulas instead of the usual induction, what part of the proof would be (slightly) simpler?

Shortening of cuts

In this subsection, we show that, under certain circumstances, the short induction implies the usual one.

Below, $\mathbb{M}$ refers to a model of weak bounded arithmetic (you can imagine theories such as $\mathsf{PV}, \forall\mathsf{PV}, \mathsf{BASIC}, \dots$).

Finally, Φ refers to a *reasonable* class of formulas; in particular, Φ should contain all quantifier-free formulas and be closed under $\wedge, \vee$.

Exercise 5. Show that the validity of induction axioms for Φ in $\mathbb{M}$ is equivalent to the statement

it is not possible to define a non-trivial cut by $\varphi \in \Phi$,

where a cut $I \subseteq \mathbb{M}$ is called **non-trivial** if I is non-empty and $I \subset \mathbb{M}$.

Definition 6. We define the cut $\mathsf{Log}(\mathbb{M})$ as $\{|m| \mid m \in \mathbb{M}\}$.

The **polynomial-induction axiom** for $\varphi(x)$ is the formula

$$(\varphi(0) \wedge \forall x(\varphi(x) \to (\varphi(2x) \wedge \varphi(2x+1))) \to \forall x \varphi(x).$$

The **length-induction axiom** for $\varphi(x)$ is the formula

$$(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(|x|).$$

Exercise 7. Show that the validity of length-induction axioms for Φ in $\mathbb{M}$ is equivalent to the statement

it is not possible to define a non-trivial short cut by $\varphi \in \Phi$,

where the cut I is **short** if it is $\subseteq \mathsf{Log}(\mathbb{M})$.

Show that the validity of polynomial-induction axioms for Φ in $\mathbb{M}$ is equivalent to the statement

it is not possible to define a non-trivial cut closed under $+, \times$ *by* $\varphi \in \Phi$.

Exercise 8. Show that the polynomial induction for Φ is equivalent to the length induction Φ.

Exercise 9. * Let A be a definable subset of $\mathbb{M}$ violating induction up to $m \in \mathbb{M}$. Show that it is possible to define a set B violating the short induction up to $\log(m)$.