The structures $K(F, G)$

The definition

Definition 1. Let $L \subseteq L_{ALL}$, we define L^2 as the language which adds a second sort of variables (sometimes called 'the set sort') $X, Y, \ldots$, which we interpret as bounded functions (and thus the $\{0,1\}$-valued functions as bounded sets). It also contains the relation $\in$ between the number sort and the set sort and the equality symbol for the set sort also denoted $=$.

Definition 2. Let $L \subseteq L_n$. The Boolean values structure $K(F, G)$ in L^2 consists of an L-closed family of function on a sample space Ω and a family G of some functions $\Theta \in \mathcal{M}$ assigning to $\omega \in \Omega$ a function $\Theta_\omega \in \mathcal{M}$ that maps a subset $\mathrm{dom}(\Theta_\omega)$ of $\mathcal{M}_n$ into $\mathcal{M}_n$.

We extend the definition of $K(F)$ by defining how $\Theta \in G$ operates on F:

$$\Theta(\alpha)(\omega) = \begin{cases} \Theta_\omega(\alpha(\omega)) & \text{if } \alpha(\omega) \in \mathrm{dom}(\Theta_\omega) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it is required that for all $\Theta \in G$ and all $\alpha \in F$: $\Theta(\alpha) \in F$.

The value of set sort equality is given by

$$[\![\Theta = \Xi]\!] = \{\omega \in \Omega; \Theta_\omega = \Xi_\omega\}/\mathcal{I},$$

and the value of the elementhood relation is

$$[\![\alpha \in \Theta]\!] = \{\omega \in \Omega; \Theta_\omega(\alpha(\omega)) = 1\}/\mathcal{I}.$$

Further we add two inductive clauses for second-order quantifiers:

$$[\![(\exists X)A(X)]\!] = \bigvee_{\Theta \in G} [\![A(\Theta)]\!]$$

$$[\![(\forall X)A(X)]\!] = \bigwedge_{\Theta \in G} [\![A(\Theta)]\!]$$

Exercise 3. Show that in $K(F, G)$ we have

$$[\![\alpha = \beta \rightarrow \Theta(\alpha) = \Theta(\beta)]\!] = 1_{\mathcal{B}}$$
$$[\![\Theta = \Xi \rightarrow \Theta(\alpha) = \Xi(\alpha)]\!] = 1_{\mathcal{B}}.$$

The structure $K(F_{bit}, G_{bit})$

Definition 4. Recall the family F_{bit} consisting of the functions

$$\begin{aligned} 0 &: \omega \mapsto 0 \\ 1 &: \omega \mapsto 0 \\ \alpha &: \omega \mapsto \omega \bmod 2 \\ \beta &: \omega \mapsto \omega + 1 \bmod 2, \end{aligned}$$

over the sample space $\Omega = \{0, \ldots, n-1\}$, where n is nonstandard and even. Let us define the family G_{bit} to obtain a structure $K(F_{bit}, G_{bit})$. Each $\Theta \in G_{bit}$ is computed by some tuple

$$\hat{\theta} = (\theta_0, \ldots, \theta_{m-1}) \in \mathcal{M}, \quad m \in \mathcal{M}_n, \quad \theta_i \in F_{bit},$$

we define for such a tuple and $\alpha \in F_{bit}$ the value $\hat{\theta}(\alpha) \in F_{bit}$ as

$$\hat{\theta}(\alpha)(\omega) = \begin{cases} \theta_{\alpha(\omega)}(\omega) & \alpha(\omega) < m \\ 0 & \text{otherwise,} \end{cases}$$

therefore each $\Theta \in G_{bit}$ induces a map $F \to F$, and we interpret any term of the form $\Theta(\alpha)$ as $\hat{\theta}(\alpha)$.

Exercise 5. The way we defined the interpretation of elements of G_{bit} is a bit off-hand. Describe for each $\Theta \in G_{bit}$ the slices Θ_ω, for each $\omega \in \Omega$.

Exercise 6. Verify that for any $\gamma \in F_{bit}$ and $\Gamma \in G_{bit}$ we have

$$\Gamma(\gamma) \in F_{bit}.$$

Remark 7. Note that the definition of G_{bit} only involved the family F_{bit} in one place, namely in the types of the elements of the tuples computing each $\Theta \in G_{bit}$. We could define generally for any family F a family $G(F)$ computed by tuples of elements from F, all the structures appearing in the book are of the form $K(F, G(F))$.

Exercise 8. Let $\Lambda \in G_{bit}$ be computed by (α, β). Find all $\gamma \in F_{bit}$ such that

$$[\![\gamma \in \Lambda]\!] = 1_{\mathcal{B}}.$$

Exercise 9. Is there $\Theta \in G_{bit}$ such that

$$\{\gamma \in F_{bit}; [\![\gamma \in \Theta]\!] = 1_{\mathcal{B}}\} = \{0, 1, \alpha, \beta\}?$$

What about $\Theta \in G_{bit}$ such that $\{\gamma \in F_{bit}; [\![\gamma \in G_{bit}]\!] = 1_{\mathcal{B}}\} = \{0, 1, \alpha\}$?

Exercise 10. Does extensionality hold in $K(F_{bit}, G_{bit})$? Namely, is

$$[\![(\forall x)(\Theta(x) = \Xi(x)) \to \Theta = \Xi]\!]?$$