Cut $\mathbb{M}_n$, language L_n, and family F_{rud}

Through the rest of this exercise sheet, n is a fixed non-standard number in $\mathbb{M}$.

Exercise 1. We define $\mathbb{M}_n$ as $\bigcup_{t\in\mathbb{M}\setminus\mathbb{N}}\{b \mid b \leq 2^{n^{1/t}}\}$. We call it a **large canonical model**. Show that $\mathbb{M}_n$ is not a substructure of the L_{ALL}-structure $\mathbb{M}$.

Exercise 2. Let $L_n \subseteq L_{\mathsf{ALL}}$ be the largest language so that $\mathbb{M}_n$ is a substructure of the L_n-structure $\mathbb{M}$. Describe this language explicitly. Is L_{PV} a subset of L_n?

Exercise 3. Is it true that any L_n-substructure of $\mathbb{M}$ is of the form $\mathbb{M}_m$ for a suitable m (note that m might differ from n)?

Is it true that any L_n-substructure of $\mathbb{M}$ is a cut?

For a language L, we define $\Sigma^b_\infty(L)$ as the class of all bounded formulas in the language L (i.e. all the quantifiers are of the form $\forall x \leq t$, or $\exists x \leq t$, where t is an L-term not containing x). The class $\forall\Sigma^b_\infty(L)$ denotes the universal closure of $\Sigma^b_\infty(L)$.

Exercise 4. Let F be any L_n-closed function family which contains all constants from $\mathbb{N}$. Let ψ be a $\forall\Sigma^b_\infty(L_n)$-sentence (i.e. no parameters from F). Show that

$$\mathbb{N} \models \psi \text{ iff } [\![\psi]\!] = 1_{\mathcal{B}}.$$

Note that there is no a priori relation between n and a sample space Ω.

We now fix a sample space Ω as all definable subsets of $[n]$ $(= \{0, \ldots, n-1\})$. A **decision tree** is defined as a codable rooted binary tree, with inner nodes labeled as queries $i \in^? \omega$ for $i \in [n]$ and two outgoing edges labeled as yes or no. A **depth** of a tree is the length of the longest path from the root. A **labeling** l of a tree T is a mapping from the leaves of T into $\mathbb{M}_n$. Finally, a labeled tree (T, l) defines a function from Ω into $\mathbb{M}_n$ with the following meaning: for $\omega \in \Omega$ let $T(\omega)$ be the unique leaf obtained by following the path from the root of T with all answers consistent with ω. The function (T, l) then maps ω to $l(T(\omega))$.

Exercise 5. We define the function family F_{rud} as containing all the function α, such that there is a labeled tree (T, l) satisfying

- $\alpha(\omega) = l(T(\omega))$ for all ω;
- the depth of T is $\leq n^{1/t}$ for some non-standard t.

Show that F_{rud} contains all the constants from $\mathbb{M}_n$ and is L_n-closed.

Exercise 6. Find a (ideally simple) function from Ω to $\mathbb{M}_n$ which is not in F_{rud}.