From truth in $\mathbb{N}$ to validity in $K(F)$

Recall the following statement

Proposition 1. Let $L \subseteq L_{\mathsf{ALL}}$ be arbitrary and F be an L-closed family. Let φ be universal L-sentence true in $\mathbb{N}$. Then, $[\![\varphi]\!] = 1_{\mathcal{B}}$. Moreover, if L contains all standard constant functions, and φ is $\exists\forall$-sentence true in $\mathbb{N}$, then $[\![\varphi]\!] = 1_{\mathcal{B}}$.

Let us now have an $L(F)$-formula $\exists y\, \varphi(\overline{x}, y)$, where φ is open. Let $f(\overline{x})$ be a function symbol not among the symbols in φ. Consider the following axiom $\mathsf{Sk}[\exists y\, \varphi(\overline{x}, y); f]$ which is defined as

$$\forall \overline{x}, y\, [\varphi(\overline{x}, y) \to \varphi(\overline{x}, f(\overline{x}))].$$

Exercise 2. Prove that $\mathsf{Sk}[\exists y\, \varphi(\overline{x}, y); f]$ implies (in predicate calculus)

$$\forall \overline{x}\, [\exists y\, \varphi(\overline{x}, y) \leftrightarrow \varphi(\overline{x}, f(\overline{x}))].$$

Exercise 3. Let us now have an $L(F)$-formula $\forall y\, \varphi(\overline{x}, y)$, where φ is open. Try to come up with a universal sentence $\mathsf{Sk}[\forall y\, \varphi(\overline{x}, y); f]$ which would imply (in predicate calculus)

$$\forall \overline{x}\, [\forall y\, \varphi(\overline{x}, y) \leftrightarrow \varphi(\overline{x}, f(\overline{x}))].$$

Exercise 4. For an $L(F)$-sentence ψ of the form $Q_1 x_1 \dots Q_k x_x\, \varphi(\overline{x})$, with φ open, iterating the above construction, try to come up with a conjuction of k universal sentences $\mathsf{Sk}[\psi; f_1, \dots, f_k]$ which would imply (in predicate calculus)

$$\psi \leftrightarrow \varphi(f_1, \dots, f_k).$$

Any tuple of functions $f_1, \dots, f_k$ from L_{ALL} which satisfies $\mathsf{Sk}[\psi; f_1, \dots, f_k]$ is called a **Skolem tuple** for ψ. For such a tuple, the open sentence $\varphi(f_1, \dots, f_k)$ is denoted ψ_{Sk}.

The main application of the above is the following

Lemma 5. Let ψ as in the previous exercise, additionally without parameters from F. Assume L contains a Skolem tuple for ψ and all standard constants. Then, for any L-closed family F it holds that

$$\mathbb{N} \models \psi \text{ iff } [\![\psi]\!] = 1_{\mathcal{B}}.$$

Exercise 6. Prove the above lemma.