Compact families and closedness under definition by cases by open formulas

Definition 1. We say F is **closed under definition by cases by open L-formulas**, if, given $\alpha_0, \alpha_1 \in F$, and $B(x)$ open L-formula without parameters and with a single free variable x, there exists $\beta \in F$ satisfying

$$\beta(\omega) = \begin{cases} \alpha_0(\omega), & \text{if } \mathbb{M} \vDash B(\alpha_0(\omega)), \\ \alpha_1(\omega), & \text{otherwise.} \end{cases}$$

Exercise 2. Show that F_{PV} is closed under definition by cases by open L-formulas for L containing only polynomial-time relations and functions.

Theorem 3. Let F be closed under definition by cases by open $L(F)$-formulas. Let $\exists x\, C(x)$ be $L(F)$-sentence. Let $\epsilon > 0$ be standard. Then, there exists $\alpha \in F$ such that

$$d([\![\exists x\, C(x)]\!], [\![C(\alpha)]\!]) < \epsilon.$$

Definition 4. We say F is **compact**, if there exists a formula $H(x, y)$ with parameters from $\mathbb{M}$, such that for

$$F_a := \{b \in \mathbb{M} \mid \mathbb{M} \vDash H(a, b)\}$$

the following conditions are satisfied

1. $\bigcap_{k \in \mathbb{N}} F_k = F$,
2. $F_k \supseteq F_{k+1}$, for all $k \in \mathbb{N}$.

Exercise 5. Argue that F_{PV} is not compact. (You don't need to prove this formally.)

However, we can have reasonable compact envelopes of F_{PV}. As an example, consider F_{subEXP} a family of functions on Ω computable by boolean circuits of size $\leq 2^{n^\xi}$, where ξ is arbitrary infinitesimal.

Exercise 6. Prove that F_{subEXP} is compact and extends F_{PV}. Can you make a tighter compact F_{PV} envelope?

The following justifies the name compact.

Theorem 7. Let F be compact. Let $C_k, k \in \mathbb{N}$ be definable sets. Assume

$$F \cap \bigcap_{l < k} C_l \neq \emptyset$$

for all k. Then,

$$F \cap \bigcap_{k \in \mathbb{N}} C_k \neq \emptyset.$$

In fact, given a countable sequence of elements $(\alpha_k)_{k\in\mathbb{N}}$ of F, and $(\alpha_i)_{i\leq t}$ its arbitrary non-standard extension in $\mathbb{M}$, there is non-standard $s \leq t$ such that

$$\alpha_i \in F, \text{ for all } i \leq s.$$

Theorem 8. Let F be a compact family closed under definition by cases by open $L(F)$-formulas. Let A be $L(F)$-sentence of the form

$$\exists x_1 \, \forall y_1 \ldots \exists x_k \, \forall y_k \, B(x_1, y_1, \ldots, x_k, y_k),$$

where B is open. Then, there are random variables $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in F$, satisfying

$$[\![A]\!] = [\![B(\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)]\!].$$