# Witnessing in definable families

**Lemma 1.** Assume $F \in \mathbb{M}$. Consider a countable antichain $\{b_k\}_{k\in\mathbb{N}}$ with $b_k = B_k/\mathcal{I}$. Let $\{\alpha_k\}_{k\in N}$ be sequence of elements of $F$.

Assume for all $k \in \mathbb{N}$, there is $\beta_k \in F$ so that for all $l \leq k$

$$\beta_k(\omega) = \alpha_l(\omega), \text{ for } \omega \in B_l \setminus \bigcup_{i<l} B_i. \tag{1}$$

Then, there exists a single $\beta \in F$, such that for all $l \leq k$

$$\beta(\omega) = \alpha_l(\omega), \text{ for } \omega \in B_l \setminus \bigcup_{i<l} B_i. \tag{2}$$

**Exercise 2.** Assuming the above lemma holds, prove that $[\![\beta = \alpha_k]\!] \geq b_k$. (This is how the above lemma is formulated in the book.)

**Exercise 3.** Prove the lemma. The fact that $F \in \mathbb{M}$ is crucial. It might also help to use two main properties of the model $\mathbb{M}$.

**Definition 4.** We say $F$ is **closed under definiton by cases**, if for any $\alpha_0, \alpha_1 \in F$, and any $B \in \mathcal{A}$, there is $\beta \in F$ such that

$$\beta(\omega) = \begin{cases} \alpha_0(\omega), & \text{if } \omega \in B, \\ \alpha_1(\omega), & \text{otherwise.} \end{cases}$$

**Exercise 5.** Show that $F$ closed under definition by cases satisfies (1) from the first lemma.

**Theorem 6.** Assume $F$ is definable and is closed under definition by cases. Let $\exists\, x C(x)$ be an $L$-sentence, possibly with parameters from $F$. Then, there exists $\alpha \in F$ such that

$$[\![\exists x\, C(x)]\!] = [\![C(\alpha)]\!].$$

**Exercise 7.** Assuming the above theorem, derive the following corollary.

**Corollary 8.** Let $F$ as above and $A$ an $L$-sentence, possibly with parameters, of the form

$$\exists x_1\, \forall y_1 \ldots \exists x_k\, \forall y_k\, B(x_1, y_1, \ldots, x_k, y_k),$$

where $B$ is arbitrary. Then, there are random variables $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in F$, satisfying

$$[\![A]\!] = [\![B(\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k)]\!].$$

(In the book, random variables $\alpha_i, \beta_i$ satisfy a subtler recursive condition.)

Consider $\Omega = \{0,1\}^n$ for $n$ non-standard. Let $L_{\mathsf{PV}}$ denote symbols for polynomial-time computable functions on $\mathbb{N}$, and $F_{\mathsf{PV}}$ denote interpretation of $L_{\mathsf{PV}}$ on $\Omega$.

**Exercise 9.** Argue that $F_{\mathsf{PV}}$ is neither definable nor is closed under definition by cases. (You don't need to prove this formally.)