Witnessing quantifiers

Exercise 1. Let A be an L-sentence of the form

$$(\exists x_1)(\forall y_1)\dots(\exists x_k)(\forall y_k)(B(x_1, y_1, \dots, x_k, y_k)).$$

Let M be an L-structure ($\{0,1\}$-valued) and consider the two player game played by the $\exists$-player and the $\forall$-player of the following form. The players alternate in picking an element $a_i \in M$ or $b_i \in M$ to substitute for the quantifier corresponding to their name. The $\exists$-player tries to make the truth value of $B(a_1, b_1, \dots, a_k, b_k)$ equal to 1 and the $\forall$-player tries to make it 0.

Show that $M \models A$ if and only if the $\exists$-player has a winning strategy.

Exercise 2. Let A be an L-sentence of the form

$$(\exists x_1)(\forall y_1)\dots(\exists x_k)(\forall y_k)(B(x_1, y_1, \dots, x_k, y_k)).$$

For any L-closed family F and consider the two player game played by the $\exists$-player and the $\forall$-player of the following form. The players alternate in picking an element $\alpha_i \in F$ or $\beta_i \in F$ to substitute for the quantifier corresponding to their name. The $\exists$-player tries to make the truth value of $[\![B(a_1, b_1, \dots, a_k, b_k)]\!]$ as close to $1_{\mathcal{B}}$ as possible and the $\forall$-player tries to make it as close to $0_{\mathcal{B}}$ as possible.

Show that there is a language L, an L-closed F and an L-sentence A such that the $\exists$-player does not have a strategy to ensure that

$$[\![B(\alpha_1, \beta_1, \dots, \alpha_k, \beta_k)]\!] = [\![(\exists x_1)(\forall y_1)\dots(\exists x_k)(\forall y_k)(B(x_1, y_1, \dots, x_k, y_k))]\!].$$

Remark 3. Assume that we can find witnesses $\alpha_1, \dots, \beta_k \in F$ such that

$$[\![(\exists x_1)(\forall y_1)\dots(\exists x_k)(\forall y_k)(B(x_1, y_1, \dots, x_k, y_k, \gamma))]\!] = [\![B(\alpha_1, \beta_1, \dots, \alpha_k, \beta_k, \gamma)]\!],$$

then we can calculate directly the value directly as

$$\{\omega \in \Omega; B(\alpha_1(\omega), \beta_1(\omega), \dots, \alpha_k(\omega), \beta_k(\omega), \gamma(\omega))/\}\mathcal{I}.$$

If we do not have such witnesses, the value may have nothing to do with

$$\{\omega \in \Omega; (\exists x_1)(\forall y_1)\dots(\exists x_k)(\forall y_k)(B(x_1, y_1, \dots, x_k, y_k, \gamma(\omega))))\}/\mathcal{I}.$$

Propositional approximations of truth values

Exercise 4. Let A be an L-sentence and F be L-closed. Then there are countably many $\alpha_k \in F$, $k \in \mathbb{N}$, such that

$$[\![(\exists x)B(x)]\!] = \bigvee_{k \in \mathbb{N}} [\![B(\alpha_k)]\!].$$

Exercise 5. Let A be an L-sentence of the form

$$(\exists x_1)(\forall y_1)\dots(\exists x_k)(\forall y_k)(B(x_1, y_1, \dots, x_k, y_k)).$$

and F be L-closed and let $\epsilon > 0$ be standard. Then there is $\ell \in \mathbb{N}$ and elements of F

$$\alpha_1^{i_1}, \beta_1^{i_1,j_1}, \alpha_2^{i_1,j_1,i_2}, \dots, \beta_k^{i_1,\dots,j_k},$$

with $i_1, \dots, j_l < \ell$ such that

$$d(\llbracket A \rrbracket, \bigvee_{i_1} \bigwedge_{j_1} \dots \bigvee_{i_k} \bigwedge_{j_k} \llbracket B(\alpha_1^{i_1}, \beta_1^{i_1,j_1}, \dots, \beta_k^{i_1,\dots,j_k}) \rrbracket) < \epsilon.$$