The models $K(F)$ continued

Definition 1. For a prefix class of formulas Γ and a theory T, let $\Gamma(T)$ be the set of all formulas from T which are in the prefix class Γ. E.g. $\forall(T)$ denotes the set of universal formulas from T.

Theorem 2. Let F be an L-closed family. Then, $\forall(\mathrm{Th}(\mathbb{N}))$ is valid in $K(F)$. Moreover, if F contains all elements of $\mathbb{N}$ as constant functions, then $\exists\forall(\mathrm{Th}(\mathbb{N}))$ is valid in $K(F)$.

Exercise 3 (*). Can we, at least in theory, prove lower bounds for lengths of proofs of some propositional proof system using a suitable model $K(F)$ and Ajtai's Argument?

Exercise 4. Find L and an L-closed family F such that $\mathrm{Th}(\mathbb{N})$ is valid in $K(F)$.

Exercise 5. Find L and an L-closed family F such that either $I\Delta_0$ or T_2 is valid in $K(F)$, but PA is not.

Exercise 6 (*). Find L and an L-closed family F such that PA is valid in $K(F)$, but $\mathrm{Th}(\mathbb{N})$ is not.

The measure of $\mathcal{B}$

Definition 7. Let $a, b \in \mathcal{B}$, we define $d(a,b) = \mu(a \triangle b)$, where $\triangle$ denotes the symmetric difference (or xor).

Exercise 8. Show that $d(a,b)$ is a metric on $\mathcal{B}$. Furthermore, show that for any $a, a', b, b' \in \mathcal{B}$:

$$d(a,b) = d(\neg a, \neg b)$$
$$d(a \wedge b, a' \wedge b') \leq d(a,a') + d(b,b')$$
$$d(a \vee b, a' \vee b') \leq d(a,a') + d(b,b')$$

Exercise 9. Let F be an L-closed family, let $A(x)$ be an open L-formula, with the only free variable x, and let $\alpha \in F$.

Show that for every standard $\epsilon > 0$: $\mathrm{Pr}_{\omega\in\Omega}[A(\alpha(\omega))] \geq \mu([\![A(\alpha)]\!]) - \epsilon$.

Exercise 10. Find L and an L-closed family F, such that for some $\gamma \in F$, we have $[\![(\forall z)A(\gamma, z)]\!] = 1_{\mathcal{B}}$ but $\mathrm{Pr}_{\omega\in\Omega}[(\forall z)A(\gamma(\omega), z)] = 0$.

Definition 11. Let $A(x,z)$ be an open L-formula. We say $\xi \in L_{all}$ is a *counter-example function for* $(\forall z)A(x,z)$, if $\mathbb{N} \models (\forall x)((\exists z)\neg A(x,z) \rightarrow \neg A(x, \xi(x)))$.

Exercise 12. Assume that $A(x,z)$ is an open L-formula, ξ is a counter-exmaple function for $\forall z A(x,z)$ and F is an L-closed family closed under ξ.

Then for any $\alpha \in F$ and standard $\epsilon > 0$, we have

$$\Pr_{\omega\in\Omega}[(\forall z)(A(\alpha(\omega), z))] \geq \mu([\![(\forall z)A(\alpha, z)]\!]) - \epsilon.$$