Definability of computations

It holds that any efficiently decidable predicate (or efficiently computable function) can be expressed as a suitable Δ_0-formula in the language L_{S_2}.

Exercise 1. * Using the above fact as a black box, show that any predicate decidable by an algorithm from PH can be expressed as a suitable Δ_0-formula in the language L_{S_2}.

We say that a bound variable is sharply bounded if the bounding quantifier is of the form

$$\forall x \leq |t| \qquad \text{or} \qquad \exists x \leq |t|$$

with t an arbitrary L_{S_2}-term.

The class of formulas with all quantifiers sharply bounded is denoted Δ_0^b.

Exercise 2. Let $\varphi(x)$ be Δ_0^b-formula. Describe an efficient algorithm that on input a decides whether $\varphi(a)$ holds.

Exercise 3. Let $\varphi(x, y)$ be Δ_0^b-formula. Assume S_2 proves

$$\forall x \exists y \varphi(x, y)$$

What can you say about the deterministic/non-deterministic complexity of an algorithm computing y given x so that $\varphi(x, y)$ holds?

Exercise 4. * What about the deterministic space complexity of the above problems?

In particular, is it possible to find an efficiently decidable predicate $P(x)$ which is impossible to express as a Δ_0^b-formula?

Definition 5. We define the class $\Sigma_1^b \subseteq \Delta_0$ as the smallest class of formulas satisfying conditions below

- $\Delta_0^b \subseteq \Sigma_1^b$
- for any $\varphi, \psi \in \Sigma_1^b$, $\varphi \wedge \psi \in \Sigma_1^b$ and $\varphi \vee \psi \in \Sigma_1^b$
- for any $\varphi(x) \in \Sigma_1^b$ and L_{S_2}-term t not containing x, $\forall x \leq |t| \varphi(x) \in \Sigma_1^b$ and $\exists x \leq |t| \varphi(x) \in \Sigma_1^b$
- for any $\varphi(x) \in \Sigma_1^b$ and L_{S_2}-term t not containing x, $\exists x \leq t \varphi(x) \in \Sigma_1^b$
- Σ_1^b closed under logical equivalence

The class Π_1^b is defined similarly with fourth conditions replaced by

- for any $\varphi(x) \in \Pi_1^b$ and L_{S_2}-term t not containing x, $\forall x \leq t \varphi(x) \in \Pi_1^b$

Exercise 6. Show that Π_1^b contains exactly formulas which are negations of the formulas from Σ_1^b.

Exercise 7. Solve exercises 2 and 3 with Δ_0^b replaced by Σ_1^b and Π_1^b.

Any efficiently decidable predicate (or efficiently computable function) can be expressed as a suitable Σ_1^b-formula in the language L_{S_2} which is a stronger statement than the one from the beginning.

However, any NP predicate or FNP relation is expressible as Σ_1^b, as well.

What makes NP different from P, or FNP different from FP?

Definition 8. Formula $\varphi(x) \in \Sigma_1^b$ is said to belong to Δ_1^b if there is a formula $\psi(x) \in \Pi_1^b$ so that

$$\forall x \varphi(x) \leftrightarrow \psi(x)$$

Formula $\varphi(x, y) \in \Sigma_1^b$ is called total if

$$\forall x \exists y \phi(x, y)$$

Given a theory T in L_{S_2}, we say that a formula $\varphi(x) \in \Sigma_1^b$ belongs to Δ_1^b in T if

$$T \vdash \forall x \varphi(x) \leftrightarrow \psi(x)$$

and similarly $\varphi(x, y) \in \Sigma_1^b$ is called total in T if

$$T \vdash \forall x \exists y \varphi(x, y)$$

Fact 9. Any efficiently decidable predicate can be expressed as a suitable Δ_1^b-formula. Any efficiently computable function can be expressed as a suitable total Σ_1^b-formula.

Theorem 10. There is a theory S_2^1 a subtheory of S_2 for which the following holds

- a predicate $P(x)$ is in P if and only if it is expressible as a formula $\varphi(x)$ which is Δ_1^b in S_2^1
- a relation $R(x, y)$ is in FP if and only if it is expressible as a formula $\varphi(x, y) \in \Sigma_1^b$ which is total in S_2^1

Exercise 11. Show that the above bullets are, in fact, equivalent. In other words, it is enough to focus only on total relations.