Boolean Algebra $\mathcal{B}$

Let Ω be infinite subset of $\mathbb{M}$ which is coded in $\mathbb{M}$. We call it a **sample space**. Let N be the size of Ω (in $\mathbb{M}$). Note that N is non-standard.

Exercise 1. Let $\mathcal{A} = \{A \mid A \in \mathbb{M}, A \subseteq \Omega\}$. Show that $\mathcal{A}$ is never a σ-algebra.

Define the **counting measure** (a.k.a. the uniform probability) on $\mathcal{A}$ as a map $A \mapsto |A|/N$, where $|\cdot|$ denotes the size function (in $\mathbb{M}$). Note that $|A|/N$ is generally a non-standard rational.

Exercise 2. We call a non-standard rational r **infinitesimal**, iff $r < 1/n$ for all $n \in \mathbb{N}$. Show that there exists a non-zero infinitesimal.

Let Ω_m denote $\{0,1\}^m$ for a non-standard m. Find a subset of $\mathcal{A}$ (over Ω_m) with non-zero infinitesimal counting measure.

Define $\mathcal{I} \subseteq \mathcal{A}$ as $A \in \mathcal{I} \iff |A|/N$ is infinitesimal.

Exercise 3. Show that $\mathcal{I}$ is an ideal. Show that it is never definable if $\mathbb{M}$.

We define a Boolean algebra $\mathcal{B}$ as $\mathcal{A}/\mathcal{I}$.

Exercise 4. We define a map μ from $\mathcal{B}$ to $\mathbb{R}$ as $\mu(A/\mathcal{I}) =$ the unique standard real infinitesimally close to $|A|/N$.

Show that μ is well-defined. Show that μ is strict, i.e. $\mu(b) > 0 \iff b \neq 0_{\mathcal{B}}$.

Remark 5. μ is above is called **Loeb's measure**.

Exercise 6. * For $\Omega = \Omega_m$ give an example of $b \in \mathcal{B}$ so that $\mu(b)$ is irrational. Is it true that any real r is a $\mu(b)$ for a suitable $b \in \mathcal{B}$? ** What about $r \in [0,1]$?

The main fact we want to prove is

Theorem 7. $\mathcal{B}$ is a complete Boolean algebra, and μ is a strict measure on $\mathcal{B}$.

Exercise 8. * Show that $\mathcal{B}$ is a σ-algebra, and μ is a measure (i.e. σ-additive).

1. Consider countable sequence $b_k \in \mathcal{B}$, where $b_k = B_k/\mathcal{I}$. We may assume $B_i \subseteq B_j$ for $i \leq j$ (explain why). Show that it is enough to find $C \in \mathcal{A}$ such that $B_k \subseteq C$ and $\lim_{n\to\infty} \mu(B_n) = \mu(C)$.

2. Show that for all standard k, there is n_k such that for all $i \geq j \geq n_k$ it holds $|B_i| \leq |B_j| + 1/k$. Thus, we can assume n_k is k.

3. Consider a non-standard sequence $(B_k)_{k\leq s} \in \mathbb{M}$ given by the property (1) of $\mathbb{M}$. Now take the following formula $\varphi(x)$

$$(x \leq s) \wedge (B_x \in \mathcal{A}) \wedge \forall i \leq x \, (B_i \subseteq B_x \wedge \frac{|B_x|}{N} \leq \frac{|B_i|}{N} + \frac{1}{i}). \qquad (1)$$

Show that there is a non-standard $l \leq s$ so that B_l satisfies (1). Prove that such B_l satisfies properties of C from 1.

Exercise 9. Show that $\mathcal{B}$ satisfies the **c.c.c. condition**, i.e. any antichain in $\mathcal{B}$ is at most countably infinite. Recall that $A \subseteq \mathcal{B}$ is called an **antichain**, iff any two $a, b \in A$ which are different, are necessary incompatible, i.e. $a \wedge b = 0_{\mathcal{B}}$. The fact that μ is strictly positive might be of help.

Exercise 10. Finish the proof of the completeness of $\mathcal{B}$ by first deriving that any family of elements $(b_i)_{i<\kappa}$ of $\mathcal{B}$ has the same set of upper bounds as one of its countable subfamilies $(b'_i)_{i\in\mathbb{N}}$.

Exercise 11. Show that infinite joins in $\mathcal{B}$ do not necesserally commute with infinite unions, i.e. $\bigvee_i (A_i/\mathcal{I}) \neq (\bigcup_i A_i)/\mathcal{I}$. Provide an example with $\bigcup_i A_i$ definable in $\mathbb{M}$.