Ambient Model

Let L_{ALL} be the language, containing all constants, relations and functions on $\mathbb{N}$, and T_{ALL} be the theory of all true L_{ALL}-sentences in $\mathbb{N}$.

We consider a non-standard model $\mathbb{M}$ of T_{ALL} satisfying the following two properties

(1) If $a_k, k \in \mathbb{N}$ is a countable family of elements of $\mathbb{M}$, then there exists a non-standard $t \in \mathbb{M}$ and a sequence $(b_i)_{i<t} \in \mathbb{M}$ so that $a_k = b_k$, for all standard k.

(2) If $A_k, k \in \mathbb{N}$ is a countable family of definable subsets of $\mathbb{M}$ so that, for each standard k, $\bigcap_{i<k} A_k \neq \emptyset$, then $\bigcap_{k\in\mathbb{N}} A_k \neq \emptyset$.

Exercise 1. In property (1) we silently assume $\mathbb{N} \subseteq \mathbb{M}$. Show that it is, indeed, the case. In other words, show that a model $\mathbb{M}$ of T_{ALL} contains $\mathbb{N}$.

Remark 2. Notation $(b_i)_{i<t} \in \mathbb{M}$ means that there exists an element of $\mathbb{M}$ coding the sequence $(b_i)_{i<t}$. This is equivalent to saying there is a binary formula (with parameters from $\mathbb{M}$) $\varphi(x, y)$ so that $\mathbb{M} \vDash \varphi(b, i) \iff i < t$ and $b = b_i$.

Remark 3. The intersection $\bigcap_{k\in\mathbb{N}} A_k$ from the property (2) need not be definable in $\mathbb{M}$.

Exercise 4. Assuming $\mathbb{M}$ as above exists, show that it is uncountable. * Is it true that $\mathbb{M}$ can have arbitrary uncountable cardinality?

Exercise 5. Show that property (1) is implied by property (2).

Exercise 6. Show that in $\mathbb{M}$ there is a number divisible by all numbers from $\mathbb{N}$ simultaneously.

Show that in $\mathbb{M}$ there is a number not divisible by any number from $\mathbb{N} \setminus \{1\}$.

Exercise 7. Assume $\varphi(x)$ is L_{ALL}-formula with parameters from $\mathbb{M}$. Assume this formula is satisfied by all elements from $\mathbb{N}$. Show that there exists a non-standard number $m \in \mathbb{M}$ which also satisfies the formula. * Is it true that $\varphi(x)$ must necessary hold for all elements of $\mathbb{M}$?

Remark 8. The previous statement is known as the *Overspill lemma*. It has a dual form (know as *Underspill lemma*): assuming $\varphi(x)$ holds for all non-standard numbers from $\mathbb{M}$, there must be a standard number n also satisfying the formula.

Exercise 9. Give an example of family A_k as in property (2) so that $\bigcap_{k\in\mathbb{N}} A_k$ is not definable in $\mathbb{M}$.

Exercise 10. ** Show that properties (1) and (2) hold in $\aleph_1$-saturated models of T_{ALL}.

Exercise 11. ** Show that there exists an $\aleph_1$-saturated model of T_{ALL}.