Proof systems NJp and NKp

Definition 1. Let $V = \{p_1, p_2, \dots\}$ be a countable set of *propositional variables*. We define a *propositional formula* in the language $\{\vee, \wedge, \to, \bot\}$ inductively:

- $\bot$ is a formula,
- any atom $p \in V$ is a formula,
- if A and B are formulas, so is $A \vee B$,
- if A and B are formulas, so is $A \wedge B$,
- if A and B are formulas, so is $A \to B$.

We use the following abbreviations:

- $\neg A := A \to \bot$,
- $\top := \neg\bot$,
- $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

Definition 2. Our proof systems NJp and NKp will work with *sequents*. The interpretation of the sequent

$$A_1, \dots, A_k \Rightarrow A$$

is

$$A_1 \wedge \cdots \wedge A_k \to A.$$

The part left of the $\Rightarrow$ is called *antecedent* and the part right of the $\Rightarrow$ is called *succedent*. The formulas in the antecedent are treated as a *multiset*.

Definition 3 (NJp)**.** The propositional proof system for intuitionistic logic NJp consists of a single axiom scheme

$$A \Rightarrow A,$$

where A is *any* formula and the following inference rules:

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B} \wedge : I, \quad \frac{\Gamma \Rightarrow A \wedge B}{\Gamma \Rightarrow A} \wedge : El, \quad \frac{\Gamma \Rightarrow A \wedge B}{\Gamma \Rightarrow B} \wedge : Er,$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \vee : Il, \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \vee : Ir, \quad \frac{\Gamma \Rightarrow A \vee B \qquad A^0, \Delta \Rightarrow C \qquad B^0, \Sigma \Rightarrow C}{\Gamma, \Delta, \Sigma \Rightarrow C} \vee : E,$$

$$\frac{A^0, \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \to B)} \to : I, \quad \frac{\Gamma \Rightarrow (A \to B) \qquad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow B} \to : E,$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow C} \bot_i.$$

The superscript ^{0} denotes that this formula may be absent in the rule and Γ, Δ and Σ denote multisets.

Proof of a sequent $\Gamma \Rightarrow A$ is a labeled tree where each node is labeled by an inference rule and each branch by the appropriate sequents, with instances of the axiom scheme appearing as leafs and which ends with the sequent $\Gamma \Rightarrow A$ at the root. Proof of a formula A is a proof of the sequent $\Rightarrow A$.

We write $\Gamma \vdash A$ is there is a proof of the sequent $\Gamma \Rightarrow A$.

Exercise 4. Derive the following "Frege axioms" in NJp:

1. $p \to (q \to p)$,
2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$.

Exercise 5. Derive the following rules:

1. Weakening: $\dfrac{\Gamma \Rightarrow A}{B, \Gamma \Rightarrow A}$,
2. Contraction: $\dfrac{B, B, \Gamma \Rightarrow A}{B, \Gamma \Rightarrow A}$,
3. Cut: $\dfrac{\Gamma \to A \qquad A, \Gamma \Rightarrow B}{\Gamma \Rightarrow B}$.

Exercise 6. Prove the deduction theorem for NJp:

$$\Gamma \vdash (A \to B) \text{ iff } A, \Gamma \vdash B.$$

Classical propositional logic

Definition 7 (NKp)**.** The propositional proof system for classical logic NKp consists of the axioms and inference rules of NJp together with an extra inference rule

$$\frac{A, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow A} \ \bot_e.$$

"Definition" 8 (Semantics for classical logic)**.** In classical logic, formulas are typically evaluated in the two-element boolean algebra on $\{0, 1\}$.

We will call any function $h : V \to \{0, 1\}$ *truth assignment* to the variables. Any such function h can be then extended into the function $\mathbf{tt}_h : \mathit{Flas} \to \{0, 1\}$ assigning truth value to all formulas according to the following rules:

- $\mathbf{tt}_h(\bot) := 0$,
- $\mathbf{tt}_h(p) := h(p)$ if $p \in V$,
- if $A = B \circ C$ where B, C are formulas and $\circ$ a logical connective, A is evaluated according to the table:

$\mathbf{tt}_h(B)$	$\mathbf{tt}_h(C)$	$\wedge$	$\vee$	$\to$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	1

We will say that formula A is a (classical) *tautology* if for any h, $\mathbf{tt}_h(A) = 1$.

Exercise 9. Check that $A \vee \neg A$ is a classical tautology.

Exercise 10. Derive the following classical tautologies in NKp and notice if you used the rule $\bot_e$.

1. $A \vee \neg A$,
2. $\neg\neg A \to A$,
3. $A \to \neg\neg A$.

Exercise 11. Do the same for the following sequents:

1. $A \to B, \neg A \to B \Rightarrow B$,
2. $A \to B, \neg A \to B \Rightarrow \neg\neg B$.

Heuristics for negation in NJp

Remember that $\neg A$ was defined as $A \to \bot$.

Exercise 12. Prove *double negation* introduction and elimination in NJp:

1. $A \to \neg\neg A$,
2. $\neg\neg\neg A \to \neg A$,
3. $\neg\neg\bot \leftrightarrow \bot$.

Exercise 13. Prove intuitionistic version of De Morgan's laws (in NJp):

1. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$,
2. $\neg(A \wedge B) \leftrightarrow (A \to \neg B)$,
3. $\neg(A \to B) \leftrightarrow (\neg\neg A \wedge \neg B)$.