The structure $K(F_{rud}, G_{rud})$

The definition

Definition 1. Recall that F_{rud} is a family containing every function

$$\alpha : \Omega \to \mathcal{M}_n$$

for which exists a labeled tree (T, l) satisyfing:

- $\forall \omega \in \Omega : \alpha(\omega) = l(T(\omega))$
- The depth of T is at most $n^{1/t}$, for some $t \in \mathcal{M} \setminus \mathbb{N}$,

where the sample space Ω is the set of definable subsets of $\{0, \dots, n-1\}$.

The family G_{rud} is then taken as the family of all Θ for which exists

$$\hat{\theta} = (\theta_0, \dots, \theta_{m-1}), m \in \mathcal{M}_n,$$

such that Θ maps any $\alpha \in F_{rud}$ to the random variable

$$\hat{\theta}(\alpha)(\omega) = \begin{cases} \theta_{\alpha(\omega)}(\omega) & \alpha(\omega) < m \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2. Show that for any $\alpha \in F_{rud}$ and $\Theta \in G_{rud}$: $\Theta(\alpha) \in F_{rud}$.

Exercise 3. Show that F_{rud} is closed under definition by open formulas in the language $L_n(F_{rud}, G_{rud})$.

Exercise 4. Show that both F_{rud} and G_{rud} are compact.

Open comprehension in $K(F_{rud}, G_{rud})$

Definition 5. For an L^2-sentence $B(\alpha_1, \dots, \alpha_k, \Theta_1, \dots, \Theta_l)$, with all parameters from F and G shown, define:

$$\langle\langle B(\alpha_1, \dots, \alpha_k, \Theta_1, \dots, \Theta_l) \rangle\rangle = \{\omega \in \Omega; B(\alpha_1(\omega), \dots, \alpha_k(\omega), \Theta_1(\omega), \dots, \Theta_l(\omega))\}.$$

Exercise 6. Let B be open L^2-formula with parameters from F or G, then:

$$[\![B]\!] = \langle\langle B \rangle\rangle / \mathcal{I}$$

Exercise 7. Let $A(y)$ be an open $L_n(F_{rud}, G_{rud})$-formula with the only free variable y.

Show that there is $t \in \mathcal{M} \setminus \mathbb{N}$ such that for all $i \in \mathcal{M}_n$ the membership in $\langle\langle A(i) \rangle\rangle$ is computed by a tree of depth bounded by $n^{1/t}$.

Exercise 8. Is the previous exercise solvable if $A(y)$ is an open $L^2_n(F_{rud}, G_{rud})$ formula?

Exercise 9. For any $\alpha \in F_{rud}$, there is $m \in \mathcal{M}_n$ such that $[\![\alpha < m]\!] = 1_{\mathcal{B}}$ and even $\langle\langle \alpha < m \rangle\rangle = \Omega$.

Definition 10. Comprehension axiom scheme for a class of formulas Γ consists of the axioms of the form $(\exists X)(\forall y < x)(y \in X \equiv A(y))$, where $A \in \Gamma$.

Exercise 11. Show that comprehension for open $L_n(F_{rud}, G_{rud})$-formulas is valid in $K(F_{rud}, G_{rud})$.