## The models $K(F)$

**Remark 1.** Assume that $\mathcal{M}$ is a model of $T_{ALL}$ satisfying the properties (1) and (2) and $\Omega$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$ are as we defined them in earlier sheets. Moreover, assume that $L \subseteq L_{all}$ is some fixed language.

**Definition 2.** We say that a non-empty set of functions $F$ is an $L$-closed family of random variables if

1. $F \subseteq \mathcal{M}$,
2. every $\alpha \in F$ is a function $\alpha : \Omega \to \mathcal{M}$,
3. $F$ contains all $L$-constants and for every $k$-ary $L$-function $f$ and every $\alpha_1, \ldots, \alpha_k$ the function

$$f(\alpha_1, \ldots, \alpha_n)(\omega) = f(\alpha_1(\omega), \ldots, \alpha_k(\omega))$$

is in $F$.

Note that $F$ itself may not be $\mathcal{M}$-definable.

**Definition 3** (The structures $K(F)$)**.** Assume that $F$ is an $L$-closed family. The structure $K(F)$ is a Boolean-valued structure, whose universe is $F$. The Boolean valuation of $L$-sentences with parameters from $F$ has values in $\mathcal{B}$ and is given by the following inductive conditions:

- $[\![\alpha = \beta]\!] = \{\omega \in \Omega; \alpha(\omega) = \beta(\omega)\}/\mathcal{I}$
- $[\![R(\alpha_1, \ldots, \alpha_k)]\!] = \{\omega \in \Omega; R(\alpha_1(\omega), \ldots, \alpha_k(\omega))\}/\mathcal{I}$, $R \in L$,
- $[\![ - ]\!]$ commutes with $\neg$, $\vee$ and $\wedge$.
- $[\![(\forall x)(A(x))]\!] = \bigwedge_{\alpha \in F} [\![A(\alpha)]\!]$
- $[\![(\exists x)(A(x))]\!] = \bigvee_{\alpha \in F} [\![A(\alpha)]\!]$.

We say an $L$-sentence $A$ is valid in $K(F)$ if $[\![A]\!] = 1_{\mathcal{B}}$.

**Remark 4.** An implication $B \to A$ is valid in $K(F)$ if and only if $[\![B]\!] \leq [\![A]\!]$.

**Fact 5.** Let $T$ be a set of $L$-sentences and $A$ an $L$-sentence. If $T \vdash A$, then there is a finite $T_0 \subseteq T$ such that

$$\bigwedge_{B \in T_0} [\![B]\!] \leq [\![A]\!].$$

**Definition 6.** Let $n \in \mathcal{M} \backslash \mathbb{N}$ be even, and $\Omega = \{m < n; m \in \mathcal{M}\}$. We let $F_{bit} = \{0, 1, \alpha, \beta\}$, where $\alpha$ is the function $\omega \mapsto \omega \bmod 2$ and $\beta$ is $\omega \mapsto \omega + 1 \bmod 2$. Also, let $L_{bit} \subseteq L_{all}$ be the language consisting of the constants 0, 1, all relations and all functions with range contained in $\{0, 1\}$.

**Exercise 7.** Show that $F_{bit}$ is $L_{bit}$-closed.

**Exercise 8.** Show that in $K(F_{bit})$ we have $[\![0 \leq \alpha \wedge \alpha \leq 1]\!] = 1_{\mathcal{B}}$.

**Exercise 9.** Show that in $K(F_{bit})$ we have

- $[\![\alpha = 0]\!] \neq 1_{\mathcal{B}}$
- $[\![\alpha = 1]\!] \neq 1_{\mathcal{B}}$
- $[\![(\alpha = 0) \vee (\alpha = 1)]\!] = 1_{\mathcal{B}}$
- $[\![\alpha = \beta]\!] = 0_{\mathcal{B}}$.

Compute the values of $\mu([\![\alpha = 0]\!])$ and $\mu([\![\alpha = 1]\!])$.

**Exercise 10** (*)**.** Show that in $K(F_{bit})$ for any $L_{bit}$-sentence $A$ we have

$$\mu([\![A]\!]) \in \{0, 1/2, 1\}.$$

**Definition 11.** We say $K(F)$ *witnesses existential sentences* if: For every quantifier free $L$-sentence $A$, potentially with $F$-parameters, we have that there is $\gamma \in F$ such that

$$[\![(\exists x)A(x)]\!] = [\![A(\gamma)]\!].$$

**Definition 12.** Let $n \in \mathcal{M} \setminus \mathbb{N}$ be even and $\Omega = \{m < n; m \in \mathcal{M}\}$. We let $F'_{bit} = \{\alpha, \beta\}$, where $\alpha$ is the function $\omega \mapsto \omega \bmod 2$ and $\beta$ is $\omega \mapsto \omega + 1 \bmod 2$. And let $L'_{bit} \subseteq L_{all}$ be the language consisting of the constants all relations, the identity function and the function $\text{not} : m \mapsto m + 1 \bmod 2$.

**Exercise 13.** Show that $F'_{bit}$ is $L'_{bit}$-closed but not $L_{bit}$-closed.

**Exercise 14.** Show that $K(F'_{bit})$ does not witness existential sentences.

**Exercise 15** (*)**.** Show that $K(F_{bit})$ does witness existential sentences.

**Exercise 16** (*)**.** Prove or disprove: Every family of random variables $F$ which has exactly one non-constant function does witness existential quantifiers.

**Definition 17.** For a prefix class of formulas $\Gamma$ and a theory $T$, let $\Gamma(T)$ be the set of all formulas from $T$ which are in the prefix class $\Gamma$. E.g. $\forall(T)$ denotes the set of universal formulas from $T$.

**Theorem 18.** Let $F$ be an $L$-closed family. Then, $\forall(\text{Th}(\mathbb{N}))$ is valid in $K(F)$. Moreover, if $F$ contains all elements of $\mathbb{N}$ as constant functions, then $\exists\forall(\text{Th}(\mathbb{N}))$ is valid in $K(F)$.

**Exercise 19** (*)**.** Can we, at least in theory, prove lower bounds for lengths of proofs of some propositional proof system using a suitable model $K(F)$ and Ajtai's Argument?

**Exercise 20.** Find $L$ and an $L$-closed family $F$ such that $\text{Th}(\mathbb{N})$ is valid in $K(F)$.

**Exercise 21.** Find $L$ and an $L$-closed family $F$ such that either $I\Delta_0$ or $T_2$ is valid in $K(F)$, but PA is not.

**Exercise 22** (*)**.** Find $L$ and an $L$-closed family $F$ such that PA is valid in $K(F)$, but $\text{Th}(\mathbb{N})$ is not.