Short Resolution proofs

Exercise 1. Write the negation of the property that every linear order on n elements has the least element as a family of propositional CNFs F_n. What is the width of the formula with respect to n?

Definition 2. We will use the following notation:

- $A_n := \bigwedge_{\substack{1 \leq i,j,k \leq n \\ i \neq j \neq k}} (\neg P_{i,j} \vee \neg P_{j,k} \vee P_{i,k})$,
- $B_n := \bigwedge_{\substack{1 \leq i,j \leq n \\ i \neq j}} (\neg P_{i,j} \vee \neg P_{j,i})$,
- $C_n := \bigwedge_{1 \leq j \leq n} \bigvee_{\substack{1 \leq i \leq n \\ i \neq j}} P_{i,j}$,

and, moreover,

- $A(i,j,k) := \neg P_{i,j} \vee \neg P_{j,k} \vee P_{i,k}$,
- $B(i,j) := \neg P_{i,j} \vee \neg P_{j,i}$,
- $C_m(j) := \bigvee_{\substack{1 \leq i \leq m \\ i \neq j}} P_{i,j}$.

Theorem 3. For $m < n$, C_m can be derived from C_{m+1}, A_n and B_n by introducing at most $O(n^2)$ new clauses.

"Proof". For $j = 1, \ldots, m$, derive $C_m(j)$ from

- $C_{m+1}(m+1)$,
- $C_{m+1}(j)$,
- $A(i, m+1, j), i \neq j$,
- $B(m+1, j)$.

How many clauses occur in this derivation for one j? □

Exercise 4. Derive an empty clause from C_2 and some clause in B_n.

Exercise 5. Using Theorem 3 and Exercise 4, what is the size of the proof of F_n in DAG-like Resolution?

Exercise 6. What is the width of this proof?

Exercise 7. Which properties of linear orders were used in the refutation?

Exercise 8. What is the property we proved (ie. the property whose negation is expressed by A_n, B_n and C_n)?

Separating DAG-like and tree-like Resolution

Definition 9. We obtain new family of CNFs G_n by replacing the clauses

$$\bigvee_{\substack{1\le i\le n\\ i\neq j}} P_{i,j}, j = 1, \dots, n$$

by the clauses

- $\neg q_{0,j}$
- $q_{i-1,j} \vee p_{i,j} \vee \neg q_{i,j}$, $i = 1, \dots, n$, $i \neq j$,
- $q_{n,j}$.

Exercise 10. Find a (tree-like) Resolution derivation of F_n from G_n.

Fact 11. $w(G_n \vdash_R \emptyset) \geq \Omega(n)$.

Theorem 12 (Ben-Sasson & Wigderson)**.** Let A be an unsatisfiable k-CNF. Then:

1. $w(A \vdash_{R^*} \emptyset) \leq k + \log(S(A \vdash_{R^*} \emptyset))$,
2. $w(A \vdash_R \emptyset) \geq k + O(\sqrt{n \log(S(A \vdash_R \emptyset))})$.

Exercise 13. Use Ben-Sasson & Wigderson to derive a lower bound on the size of tree-like Resolution proof of G_n.

Exercise 14. Use Exercise 5 and 10 to conclude that G_n has a proof in DAG-like Resolution of polynomial size.