1 Algebraic Models

Definition 1. **Lattice** is a poset $\mathcal{P} = (P, \leq)$ such there for all $a, b \in L$, $\sup\{a, b\}$ and $\inf\{a, b\}$ exists. We define two new binary operations $\vee$ and $\wedge$ on P as:

- $a \vee b := \sup\{a, b\}$,
- $a \wedge b := \inf\{a, b\}$.

We say that lattice is **bounded** is it has the greatest the least element (denoted by 1 or $\top$ and 0 or $\bot$).

Exercise 2. Another way to define a lattice is to say it is a model of a theory in the language $(\vee, \wedge)$ with the axioms being the universal closure of the following:

1. $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *(associativity)*,
2. $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ *(commutativity)*,
3. $a \vee a = a$, $a \wedge a = a$ *(idempotency)*,
4. $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$ *(absorption laws)*.

How would you define a $\leq$ relation on such structure $\mathcal{L}$ so it satisfies the (first) definition above?

Definition 3. **Heyting algebra** (or pseudo-Boolean algebra) $\mathcal{H} = (H, \leq)$ is a bounded lattice such that for every $a, b \in H$, $\max\{c | a \wedge c \leq b\}$ exists. we define a new binary operation $\to$ (called *pseudocomplement*) as

$$a \to b := \max\{c | a \wedge c \leq b\}.$$

Exercise 4. Show that

1. $\bot \to \bot = \top$,
2. $a \to b = \top$ iff $a \leq b$,
3. $(a \wedge (a \to b)) \to b = \top$,
4. $((c \wedge a) \to b) \to (c \to (a \to b)) = \top$.

Definition 5. A valuation from the set of propositional variables $\mathcal{P}$ to a Heyting algebra $\mathcal{H}$ is a function $\tilde{h} : \mathcal{P} \to \mathcal{H}$. The valuation extends into a valuation from propositional formulas to $\mathcal{H}$ as a homomorphism:

1. $h(p) := \tilde{h}(p)$,
2. $h(\top) := \top$, $h(\bot) := \bot$
3. $h(\varphi \vee \psi) := h(\varphi) \vee h(\psi)$,
4. $h(\varphi \wedge \psi) := h(\varphi) \wedge h(\psi)$,
5. $h(\varphi \to \psi) := h(\varphi) \to h(\psi)$.

Formula φ is **true** in a Heyting algebra $\mathcal{H}$ and a valuation h if $h(\varphi) = \top$. Formula φ is **valid** in $\mathcal{H}$ if it is true with respect to any valuation h.

Formula φ is **valid** if it is valid in any Heyting algebra $\mathcal{H}$. Sequent $\Gamma \Rightarrow \Delta$ is valid if the formula $\bigwedge_{\varphi \in \Gamma} \varphi \to \bigvee_{\psi \in \Delta} \psi$ is valid.

Lemma 6. LJpm is sound.

Exercise 7. Show soundness for your favourite rule of LJpm.

Definition 8. Define

$$\hat{\varphi} := \{\psi | (\varphi \leftrightarrow \psi) \text{ is derivable}\}.$$

The **Lindenbaum algebra** $\mathcal{B}$ is a Heyting algebra defined as

- $B := \{\hat{\varphi} | \varphi \text{ a formula}\}$,
- $\hat{\varphi} \vee \hat{\psi} := (\widehat{\varphi \vee \psi})$,
- $\hat{\varphi} \wedge \hat{\psi} := (\widehat{\varphi \wedge \psi})$.

Exercise 9. Verify that

1. $\hat{\varphi} \to \hat{\psi} = (\widehat{\varphi \to \psi})$,
2. $(\widehat{p \to p}) = \top$,
3. $(\widehat{p \wedge \neg p}) = \bot$.

Theorem 10. LJpm is complete.