Kripke Semantics II

Inuitionistic Sequent Calculus

Definition 1 (The Gentzen-type Sequent Calculus $LJpm$)**.** The system $LJpm$ works with sequents $\Gamma \Rightarrow \Delta$ which contain a multi-set of formulas both in the antecedent and the succedent. The sequent $\Gamma \Rightarrow \Delta$ has the same semantical meaning as $\bigwedge_{\gamma \in \Gamma} \gamma \to \bigvee_{\delta \in \Delta} \delta$. The system has the following rules:

- Axioms: $p \Rightarrow p$ and $\bot \Rightarrow p$ for an atom p.
- Logical inference rules:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi \to \psi \Rightarrow \Delta, \varphi \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

- Structural inference rules:

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}(\text{contr})\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}(\text{weak})\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

Note that the $\Rightarrow\to$ rule does not have Δ in its premises. Moreover, if $\neg$ were taken to be a separate connective, the corresponding rules become:

$$\frac{\varphi, \Gamma \Rightarrow}{\Gamma \Rightarrow \Delta, \neg\varphi} \qquad \frac{\neg\varphi, \Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$$

Another rule, which is not part of $LJpm$ but we will consider it later is the cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Exercise 2. Prove $\Rightarrow p \wedge q \to q \wedge p$ and $p, p \to q \Rightarrow q$ in $LJpm$.

Exercise 3. Show that every rule of $LJpm$ has the subformula property: Each formula in the premise is a subformula of some formula in the conclusion. Every $LJpm$-proof consists of subformulas of its last sequent.

Lemma 4. For a sequent $\Gamma \Rightarrow \Delta$ the following are equivalent:

- $\Gamma \Rightarrow \Delta$ is provable in $LJpm$ with cut.
- $\Gamma \Rightarrow \bigvee_{\delta \in \Delta} \delta$ is provable in NJp.

Moreover:

1. Every introduction rule is the succedent logical rule of $LJpm$ for the same connective.

2. Every elimination rule can be translated by the antecedent rules for the same connective plus cut.

3. Translation of every rule of $LJpm$ is provable in NJp.

Definition 5. The value of a multi-succedent sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ in a world $w \in W$ of a model $\langle W, R, V \rangle$ is given as:

$V(\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n, w) = 1$ iff for every w', $R(w, w')$ we have that $V(\alpha_1, w') = \cdots = V(\alpha_m, w') = 1$ implies $V(\beta_j, w')$ for some $j \leq n$.

Exercise 6. Show that $LJpm$ plus cut is sound.

Exercise 7. A rule if invertible if the derivability of the conclusion implies the derivability of the premises. Show that all rules of $LJpm$, except for the succdent rules for $\neg$ and $\rightarrow$ are invertible.

Exercise 8. Show that $\Rightarrow \neg$ and $\Rightarrow \rightarrow$ are not invertible.

Definition 9. A sequent $\Gamma \Rightarrow \Delta$ is falsified in a world w of a model $\langle W, R, V \rangle$ iff $V(\bigwedge \Gamma, w) = 1$ and $V(\bigvee \Delta, w) = 0$. This implies $V(\Gamma \Rightarrow \Delta, w) = 0$.

Exercise 10. Find a Kripke frame with a world w such that $V(\Gamma \Rightarrow \Delta, w) = 0$ but w does not falsify $\Gamma \Rightarrow \Delta$.

Completeness and Admissibility of Cut

Definition 11. In this sheet, $\Gamma \vdash \Delta$ means that $\Gamma \Rightarrow \Delta$ is derivable in $LJpm$.

Definition 12. $Sub(\Gamma)$ stands for the set of all subformulas of Γ. A sequent $\Gamma \Rightarrow \Delta$ is *complete* if it is unprovable and for any $\varphi \in Sub(F, \Delta)$ either $\varphi \in \Gamma \cup \Delta$ or

$$\Gamma \vdash \Delta, \varphi \text{ and } \varphi, \Gamma \vdash \Delta,$$

that is both sequents $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ are derivable.

Definition 13. A sequent $\Gamma \Rightarrow \Delta$ is *saturated for invertible rules* if it is underivable and the following conditions are satisfied for any φ and ψ:

- $(\Rightarrow \wedge)$: $(\varphi \wedge \psi) \in \Delta$ implies $\varphi \in \Delta$ or $\psi \in \Delta$
- $(\wedge \Rightarrow)$: $(\varphi \wedge \psi) \in \Gamma$ implies $\varphi \in \Gamma$, $\psi \in \Gamma$
- $(\Rightarrow \vee)$: $(\varphi \vee \psi) \in \Delta$ implies $\varphi \in \Delta$, $\psi \in \Delta$
- $(\vee \Rightarrow)$: $(\varphi \vee \psi) \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$
- $(\rightarrow\Rightarrow)$: $(\varphi \rightarrow \psi) \in \Gamma$ implies $\varphi \in \Delta$ or $\psi \in \Gamma$.

Lemma 14 (Saturation)**.** If $\Gamma \Rightarrow \Delta$ is complete, then it is saturated for invertible rules.

Lemma 15 (Completion)**.** Any underivable sequent $\Gamma_0 \Rightarrow \Delta_0$ can be extended to a complete sequent consisting of subformulas of Γ_0, Δ_0.

Definition 16. We define the Kripke model: $\mathbf{K} = \langle W, R_{\subseteq}, V_{\in} \rangle$:

- W is the set of all complete sequents.
- $R_{\subseteq}(\Gamma \Rightarrow \Delta, \Gamma' \Rightarrow \Delta')$ iff $\Gamma \subseteq \Gamma'$
- $V_{\in}(p, \Gamma \Rightarrow \Delta) = 1$ iff $p \in \Gamma$

Definition 17. A set of sequents M is saturated for non-invertible rules if the following condition is satisfied for every $\Gamma \Rightarrow \Delta$ in M:

($\Rightarrow\rightarrow$): if $(\varphi \rightarrow \psi) \in \Delta$, then there is a sequent $\Gamma' \Rightarrow \Delta'$ in M such that $\varphi, \Gamma \subseteq \Gamma'$ and $\psi \in \Delta'$.

A set of sequents M is saturated if it is saturated for non-invertible rules and every sequent in M is saturated for invertible rules.

Lemma 18. The set W of all complete sequents is saturated.

Definition 19. A model defined by a saturated set M is $\mathbf{K}_M = \langle M, R_m, V_m \rangle$, where $R_{\mathbf{M}} = R_{\subseteq}$, $V_M = V_{\in}$ as above.

Theorem 20. Let M be a saturated set. Then for $w \equiv \Gamma \Rightarrow \Delta, m \in M$:

$$\begin{aligned} &\theta \in \Gamma \text{ implies } V_M(\theta, w) = 1 \\ &\theta \in \Delta \text{ implies } V_M(\theta, w) = 0 \\ &V_M(\Gamma \Rightarrow \Delta, w) = 0. \end{aligned}$$

Corollary 21. Each sequent unprovable in $LJpm$ is falsified in the canonical model **K**. Hence every valid sequent is provable in $LJpm$.

Moreover, every unprovable sequent $\Gamma \Rightarrow \Delta$ is falsified in a finite model $\mathbf{K}_m$, where $M = M_{\Gamma \Rightarrow \Delta}$ is the set of all complete sequents consisting of subformulas of Γ, Δ. The number of elements of $M_{\Gamma \Rightarrow \Delta}$ is bounded by 4^s, where s is the number of subformulas of Γ, Δ.

Theorem 22 (Soundness and completeness)**.** A formula is provable in $LJpm$ iff it is valid.

Moreover, a formula is provable in $LJpm$ iff it is valid in all finite pointed models where accessibility relation is a partial order.

Corollary 23. Cut rule is admissible in $LJpm$: If $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ are provable, then $\Gamma \Rightarrow \Delta$ is provable.

Corollary 24. $LJpm$ is equivalent to NJp with respect to derivability of sequents $\Gamma \Rightarrow \alpha$.