Kripke Semantics

Kripke Frames

Remark 1. Our goal in this section is to model the semantics for intuitionistic logic by interpreting *true* as something firmly established, and *false* as something "not yet true" in a dynamic situition, where we can learn more and more information about the propositional atoms.

Definition 2. A propositional intuitionistic model (intuitionistic Kripke model) is a triple $\langle W, R, V \rangle$, where W is a non-empty set whose elements are called *worlds*, R is a binary reflexive and transitive relation on f called the *accessibility relation* and V is a function assigning to every propositional variable p and every world p a value $V(p, w) \in \{0, 1\}$ such that it is monotone with respect to R:

$$\text{if } R(w, w') \text{ and } V(p, w) = 1, \text{ then } V(p, w') = 1$$

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For a particular w and an arbitrary formula φ we define the truth value $V(\varphi, w) \in \{0, 1\}$ inductively:

- $V(\bot, w) = 0$
- $V(\alpha \wedge \beta, w) = \min(V(\alpha, w), V(\beta, w))$
- $V(\alpha \vee \beta, w) = \max(V(\alpha, w), V(\beta, w))$
- $V(\alpha \to \beta, w) = 1$ iff for all w' such that $R(w, w')$ we have that $V(\alpha, w') = 1$ implies $V(\beta, w') = 1$

If $V(\varphi, w) = 1$ in a model $M = \langle W, R, V \rangle$ we write $M, w \models \varphi$. If $M, w \models \varphi$ for every world w, then we write $M \models \varphi$. We write $\models \varphi$ if $M \models \varphi$ for every model M. The tuple $\langle W, R \rangle$ is called a *Kripke frame*.

Exercise 3. Find a two world propositional intuitionistic model $\langle W, R, V \rangle$ and a world w in it such that $V(p \vee \neg p, w) = 0$.

Exercise 4. Find a two world propositional intuitionistic model $\langle W, R, V \rangle$ and a world w in it such that $V(\neg\neg p \to p, w) = 0$.

Exercise 5. Find a three world propositional intuitionistic model $\langle W, R, V \rangle$ and a world w in it such that $V(\neg p \vee \neg\neg p, w) = 0$.

Lemma 6 (Monotonicity)**.** Let $\langle W, R, V \rangle$ be a propositional intuitionistic model, then for any $w, w' \in W$ and a formula φ:

$$R(w, w') \text{ and } V(\varphi, w) = 1 \text{ imply } V(\varphi, w') = 1.$$

Definition 7 (Valuation of sequents)**.** We define $C(\alpha_1, \ldots, \alpha_m \Rightarrow, w) = 1$ iff for any w' such that $R(w, w')$ we have

$$V(\alpha_1, w') = V(\alpha_2, w') = \cdots = V(\alpha_k, w') = 1 \text{ implies } V(\beta, w') = 1.$$

Lemma 8. All the rules of NJp are sound: If all the premises are true is a world w of a Kripke model, then the conclusion is also true in w.

Definition 9 (Pointed models)**.** A pointed intuitionistic propositional model is a tuple $M = \langle G, W, R, V \rangle$, where $\langle W, R, V \rangle$ is a propositional intuitionistic model and $G \in W$. We define $M \models \varphi$ to be true iff $V(\varphi, G) = 1$. The tuple $\langle G, W, R \rangle$ is called a *pointed frame*.

Lemma 10. A formula is valid iff it is valid in every pointed intuitionistic propositional model.

Lemma 11. A formula is valid iff it is valid in every pointed intuitionistic propositional model whose accessibility relation is a partial order. That is, it is also anti-symmetric: if $R(w, w')$ and $R(w', w)$ then $w = w'$.

Frame Conditions

Lemma 12 (Frame condition for $p \vee \neg p$)**.** Let $F = \langle G, W, R \rangle$ where every $w \in W$ is accessible from G. Show that $p \vee \neg p$ is valid on every model over the frame F iff F is total, that is for every $w, w' \in W$ we have $R(w, w')$.

Exercise 13. Show that $p \vee \neg p$ is valid in every model over a frame, iff R is symmetric.

Exercise 14. Characterize those pointed frames for which $(p \to q) \vee (q \to p)$ is valid in every model over them.

Exercise 15. Characterize those pointed frames for which $\neg p \vee \neg\neg p$ is valid in every model over them.