The BHK Interpretation

Assignment $\mathcal{T}$

Remark 1. Our goal is to formalize the idea of *realizability.* The idea is to have a notion of *contructions* such that every intuitionistic tautology will obtain a construction which 'realizes' it. The idea is to have a relation $c\mathbf{r}\alpha$ read as 'c realizes α' with the following properties:

$$c\mathbf{r}(\alpha_0 \wedge \alpha_1) \text{ iff } c = (a_0, a_0) \text{ and } a_i \text{ realizes } \alpha_i \text{ for } i \in \{0,1\}$$
$$c\mathbf{r}(\alpha_0 \vee \alpha_1) \text{ iff } c = (i, a) \text{ and } a \text{ realizes } \alpha_i$$
$$c\mathbf{r}(\alpha \to \beta) \text{ iff } c \text{ is a function such that for every } d\text{: if } d\mathbf{r}\alpha\text{, then } c(d)\mathbf{r}\beta$$
$$\text{not } c\mathbf{r} \perp$$

Exercise 2. Show that for any relation $\mathbf{r}$ satisfying the properties above, we have: $c\mathbf{r}\neg\alpha$ iff for every d: $\neg d\mathbf{r}\alpha$.

Exercise 3. If we understand construction as (computer) programs, try to describe the instructions of constructions realizing the following formulas:

- $(\alpha_0 \vee \alpha_1) \to (\beta_0 \vee \beta_1)$
- $\alpha \to (\beta \wedge \gamma)$
- $(\alpha_0 \vee \alpha_1) \to \beta \wedge (\beta_0 \vee \beta_1)$
- $\alpha \to ((\alpha_0 \vee \alpha_1) \to (\beta_0 \vee \beta_1))$

Definition 4 (The assignment $\mathcal{T}$)**.** The construction we will use will be terms consisting of the following basic constructs:

- Countably many variables $x^\varphi, y^\varphi, z^\varphi, \ldots$, for every formula φ.
- A pairing function $\mathbf{p}$ and its projections $\mathbf{p}_i$ satisfying $\mathbf{p}_i(\mathbf{p}(a_0, a_1)) = a_i$ for $i \in \{0,1\}$.
- Lambda abstraction: $(\lambda x.t)$ interpreted as: $(\lambda x.t)(u) = t[x/u]$, substituting the term u for every free occurance of x in t.
- Injections into a disjoint unions: $\mathbf{k}_0$ and $\mathbf{k}_1$.
- Case distinction for disjunction:

$$D_{x_0,x_1}(\mathbf{k}_i t, t_0, t_1) = t_i[x_i/t_i],\ i \in \{0,1\}$$

- A function $\perp_\varphi$ providing a trivial realization of φ under the assumption a contradiction was obtained.

We will now define a map $\mathcal{T}$ which to every NJp proof d of a sequent $\Gamma \Rightarrow \varphi$ assign a term u which under the assumptions Γ realizes φ under the BHK interpretation. The map $\mathcal{T}$ simply replaces every rule in the proof d by an appropriate assignment rule to obtain the final term (we use the notation $\mathbf{z} : \Gamma$ to mean $z^{\gamma_1} : \gamma_1, \ldots, z^{\gamma_k} : \gamma_k$ giving variable names to the assumed realization of formulas in Γ, this is sometimes called *context*):

Axioms:

$$x : \varphi \Rightarrow x : \varphi$$

Inference rules:

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : \varphi \quad \mathbf{z}' : \Delta \Rightarrow u : \psi}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta \Rightarrow \mathbf{p}(t, u) : (\varphi \wedge \psi)}(\wedge I) \qquad \frac{\mathbf{z} : \Gamma \Rightarrow t : (\varphi_1 \wedge \varphi_2)}{\mathbf{z} : \Gamma \Rightarrow \mathbf{p}_i(t) : \varphi_i}(\wedge E),\ i \in \{0, 1\}$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : \varphi_i}{\mathbf{z} : \Gamma \Rightarrow \mathbf{k}_i t : (\varphi_0 \vee \varphi_1)}(\vee I),\ i \in \{0, 1\}$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : (\varphi \vee \psi) \quad (x^{\varphi} : \varphi)^0, \mathbf{z}' : \Delta \Rightarrow t_0 : \theta \quad (y^{\psi} : \psi), \mathbf{z}'' : \Sigma \Rightarrow t_1 : \theta}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta, \mathbf{z}'' : \Sigma \Rightarrow D_{x,y}(t, t_0, t_1) : \theta}(\vee E)$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : (\varphi \to \psi) \quad \mathbf{z}' : \Delta \Rightarrow u : \psi}{\mathbf{z} : \Gamma, \mathbf{z}' : \Delta \Rightarrow t(u) : \psi}(\to E) \qquad \frac{(x : \varphi)^0, \mathbf{z} : \Gamma \Rightarrow t : \psi}{\mathbf{z} : \Gamma \Rightarrow (\lambda x.t) : (\varphi \to \psi)}(\to I)$$

$$\frac{\mathbf{z} : \Gamma \Rightarrow t : \bot}{\mathbf{z} : \Gamma \Rightarrow \bot_{\varphi}(t) : \varphi}(\bot_i)$$

Exercise 5. For each of the following formulas find a term which realizes it:

- $p \to p$
- $p \to (q \to p)$
- $p_i \to (p_0 \vee p_1),\ i \in \{0, 1\}$
- $(p \to q) \to ((q \to r) \to (p \to r))$
- $(p \to q) \to ((q \to r) \to (p \vee q \to r))$

Remark 6. There is an inverse map $\mathcal{D}$ which restores an NJp proof from every deductive term. The fact that $\mathcal{T}$ and $\mathcal{D}$ are mutual inverses is called the Curry–Howard isomorphism.