Glivenko's theorem and negative translation

Direct chaining and analysis into subgoals

The problem of deriving given sequent (goal) S can often be reduced to deriving simpler sequents (subgoals), say S_1, S_2, if the rule

$$\frac{S_1 \qquad S_2}{S}$$

is derivable. Let us derive some of these rules.

Exercise 1. (Analysis into subgoals) Derive the following rules in NJ_p:

$$\frac{\alpha, \Gamma \implies \beta \qquad \beta, \Gamma \implies \alpha}{\Gamma \implies \alpha \leftrightarrow \beta}, \quad \frac{\alpha, \Gamma \implies \varphi \qquad \beta, \Gamma \implies \varphi}{\alpha \vee \beta, \Gamma \implies \varphi},$$
$$\frac{\alpha^0, \Gamma \implies \bot}{\Gamma \implies \neg\alpha}$$

Exercise 2. (Direct chaining) Derive the following rules in NJ_p:

$$\frac{\Gamma \implies \alpha \leftrightarrow \beta}{\Gamma \implies \alpha \longrightarrow \beta}, \quad \frac{\Gamma \implies \alpha \leftrightarrow \beta}{\Gamma \implies \beta \longrightarrow \alpha}, \quad \frac{\Gamma \implies \bot}{\Gamma \implies \beta},$$
$$\frac{\Gamma \implies \neg\alpha \qquad \Delta \implies \alpha}{[\Gamma, \Delta] \implies \beta}$$

Definition 3. A deduction using only rules in Exercise 2, cut and structural rules is called direct chaining.

Definition 4. (ADC) The combination of Analysis into subgoals and Direct Chaining is called ADC-method or simply ADC. It is not complete: some valid formulas are not deducible using ADC.

Question 5. Estimate the complexity of ADC and find a suitable subclass for which it is complete.

Heuristics for negation

Here we derive some properties of negation used later. Recall that $\neg\alpha \equiv (\alpha \to \bot)$ and $\Gamma \vdash \alpha$ if and only if the sequent $\Gamma \implies \alpha$ is derivable in NJ_p.

Exercise 6. 1. $\Gamma \vdash \neg\alpha$ iff $\Gamma, \alpha \vdash \bot$

2. double negation introduction and elimination:

   (a) $\vdash \alpha \to \neg\neg\alpha$

   (b) $\vdash \neg\neg\neg\alpha \leftrightarrow \neg\alpha$

   (c) $\vdash \neg\neg\bot \leftrightarrow \bot$

3. de Morgan's laws valid intuitionistically:

(a) $\vdash \neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$

(b) $\vdash \neg(\alpha \wedge \beta) \leftrightarrow (\alpha \rightarrow \neg\beta)$

(c) $\vdash \neg(\alpha \rightarrow \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\beta)$

4. $\Gamma \vdash \bot$ iff $\Gamma \vdash \alpha$, $\Gamma \vdash \neg\alpha$ for some α

Glivenko's theorem

Theorem 7. (Glivenko's theorem) $\Gamma \vdash \neg\alpha$ if and only if $\bigwedge\Gamma \rightarrow \neg\alpha$ is a tautology. In particular, a formula beginning with a negation is derivable in NJ_p if and only if it is a tautology.

Embedding of NK_p into NJ_p

The last section shows that it is possible to embed classical logic NK_p into intuitionistic system NJ_p by inserting double negation to turn off constructive content of disjunctions and atomic formulas (which stand for arbitrary sentences and may potentially have constructive content).

Definition 8. Define inductively operation ^{neg} transforming formulas into formulas:

1. $p^{neg} := \neg\neg p$ for atomic p
2. $(\alpha \wedge \beta)^{neg} := \alpha^{neg} \wedge \beta^{neg}$
3. $(\alpha \rightarrow \beta)^{neg} := \alpha^{neg} \rightarrow \beta^{neg}$
4. $(\alpha \vee \beta)^{neg} := \neg\left(\neg\alpha^{neg} \wedge \neg\beta^{neg}\right)$

In connection with the last clause, note that de Morgan's law 3a in 6 implies that:

$$\vdash \neg(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\neg(\alpha \vee \beta).$$

Note also that:

$$\vdash (\neg\alpha)^{neg} \leftrightarrow \neg\left(\alpha^{neg}\right),$$

since $(\neg\alpha)^{neg} \equiv (\alpha^{neg} \rightarrow \neg\neg\bot) \leftrightarrow (\alpha^{neg} \rightarrow \bot)$.

Definition 9. A propositional formula is negative if it does not contain $\vee$, and all atomic subformula are negated.

Exercise 10. Prove the following:

1. $\vdash \neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$, $\quad \vdash \neg\neg(\alpha \rightarrow \beta) \leftrightarrow (\neg\neg\alpha \rightarrow \neg\neg\beta)$
2. $\vdash \neg\neg\alpha \leftrightarrow \alpha$ for every negative formula α.

Exercise 11. Prove that every rule

$$\frac{\Gamma \Longrightarrow \varphi \qquad \Delta \Longrightarrow \psi}{\Sigma \Longrightarrow \theta}$$

of NK_p is stable under Gödel's negative translation. That is, the rule

$$\frac{\Gamma^{neg} \implies \varphi^{neg} \qquad \Delta^{neg} \implies \psi^{neg}}{\Sigma^{neg} \implies \theta^{neg}}$$

is derivable in NJ_p and similarly for one-premise and three-premise rules.

Theorem 12. A sequent $\Gamma \implies \Delta$ is derivable in NK_p if and only if $\Gamma^{neg} \implies \Delta^{neg}$ is derivable in NJ_p.