Herbrand's theorem, Student-Teacher computations and Finite axiomatizibility

Easy witnessing for PV_1

Exercise 1. Let T be an universal L-theory. Show that for $M \models T$ and $N \subseteq M$ a substructure, we have $N \models T$.

Theorem 2 (Herbrand)**.** Let T be a universal L-theory, let $\varphi(x, y)$ an open L-formula. If

$$T \vdash (\forall x)(\exists y)\varphi(x, y),$$

then there exist L-terms $t_1,\ldots,t_k$ such that

$$T \vdash (\forall x) \bigvee_{i=1}^{k} \varphi(x, t_i(x)).$$

Exercise 3. Prove Herbrand's theorem.

Possible proof via midsequent lemma.

Fact 4 (PV-symbols are closed under definition by cases)**.** Let $f_1(x), \ldots, f_k(x)$, $g_1(x), \ldots, g_k(x)$ and $h_1(x),\ldots,h_k(x)$ be PV-symbols. Then there is a PV symbol $f(x)$, such that PV_1 proves

$$f(x) = \begin{cases} f_1(x) & \text{if } g_1(x) = h_1(x) \\ f_2(x) & \text{else if } g_2(x) = h_2(x) \\ \vdots & \\ f_k(x) & \text{else if } g_k(x) = h_k(x) \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 5. Show that if for φ open PV-formula

$$\mathrm{PV}_1 \vdash (\forall x)(\exists y)\varphi(x, y),$$

then there is a PV-symbol f, such that

$$\mathrm{PV}_1 \vdash (\forall x)\varphi(x, f(x)).$$

The theories S_2^i and T_2^i

The complexity classes Σ_i^p (respectively Π_i^p) consist of decisions problems computable by alternating Turing machine in polynomial time which starts in existential (respectively universal) guessing mode and can toggle it $(i-1)$-many times. Notice that $\Sigma_1^p = \mathbf{NP}$ and $\Pi_1^p = \mathbf{coNP}$. The class $\mathbf{PH} = \bigcup_{i=1}^{\infty} \Sigma_i^p$, it is generally expected that $\mathbf{PH} \neq \Sigma_i^p$ for any i, this is sometimes described as "polynomial hierarchy not collapsing".

Definition 6. L_{S_2}-formula φ is Σ_i^b if it is bounded and after removing all sharply bounded quantifiers the first quantifier on each path through the tree representing the formula is existential bounded and the following quantifiers change from existential to universal and back at most $i-1$ many times.

L_{S_2}-formula φ is Π_i^b if it satisfies the conditions for Σ_i^b, but instead the first quantifier is universal bounded on each path.

Exercise 7. Find an L_{S_2}-formula in Σ_0^b, Π_2^b and one which is not in $\Sigma_1^b \cup \Pi_1^b$ but in $\Sigma_2^b \cap \Pi_2^b$.

Fact 8. For $i \geq 1$: $\Sigma_i^b(\mathbb{N}) = \Sigma_i^p$ and $\Pi_i^b(\mathbb{N}) = \Sigma_i^p$.

Definition 9. For $i \geq 0$ we define

$$\begin{aligned} S_2^i &= BASIC + \Sigma_i^b\text{-PIND} = BASIC + \Pi_i^b\text{-PIND} \\ T_2^i &= BASIC + \Sigma_i^b\text{-IND} = BASIC + \Pi_i^b\text{-IND}, \end{aligned}$$

additionally $S_2 = \bigcup_{i=1}^{\infty} S_2^i$ and $T_2 = \bigcup_{i=1}^{\infty} T_2^i$.

Fact 10. For $i \geq 1$: $S_2^i \subseteq T_2^i \subseteq S_2^{i+1}$.

Exercise 11. Show that $S_2 = T_2$.

Intermezzo about Sharply bounded aritmetics

Fact 12 (Takeuti, 1987)**.**

$$S_2^0 \nvdash \text{“The predecessor function is total.”}$$

Fact 13 (Jeřábek, 2006)**.** PV_1 is conservative over T_2^0 with the language expanded by the function

$$MSP(x, y) = \lfloor x/2^y \rfloor,$$

and its defining axioms.

Fact 14 (Boughattas, Kołodziejczyk, 2009)**.**

$$T_2^0 \nvdash \text{“Every nontrivial divisor of a power of 2 is even.”}$$

Corollary 15. $S_2^0 \nsubseteq T_2^0$ and $T_2^0 \nsubseteq S_2^0$.

Finite axiomatizibility

Fact 16. For $i \geq 1$ the theories S_2^i and T_2^i are finitely axiomatizable.

Exercise 17. Show that for some $i \geq 1$ we have $T_2^i = T_2$ if and only if T_2 is finitely axiomatizable.

Theorem 18 (KPT witnessing)**.** Let T be a universal L-theory, let $\varphi(x, y)$ an open L-formula. If

$$T \vdash (\forall x)(\exists y)(\forall z)\varphi(x, y, z),$$

then there exist L-terms $t_1, \ldots, t_k$ such that

$$T \vdash (\forall x) \bigvee_{i=1}^{k} \varphi(x, t_i(x, c_1, \ldots, c_{i-1}), c_i).$$

Theorem 19 (Krajíček–Pudlák–Takeuti,1990)**.** If

$$T_2^i \nvdash \text{“Polynomial hierarchy collapses.”},$$

then

$$T_2^i \neq T_2.$$