## Definability of computations

It holds that any efficiently decidable predicate (or efficiently computable function) can be expressed as a suitable  $\Delta_0$ -formula in the language  $L_{S_2}$ .

**Exercise 1.** \* Using the above fact as a black box, show that any predicate decidable by an algorithm from PH can be expressed as a suitable  $\Delta_0$ -formula in the language  $L_{S_2}$ .

We say that a bound variable is sharply bounded if the bounding quantifier is of the form

$$\forall x \le |t|$$
 or  $\exists x \le |t|$ 

with t an arbitrary  $L_{S_2}$ -term.

The class of formulas with all quantifiers sharply bounded is denoted  $\Delta_0^b$ .

**Exercise 2.** Let  $\varphi(x)$  be  $\Delta_0^b$ -formula. Describe an efficient algorithm that on input a decides whether  $\varphi(a)$  holds.

**Exercise 3.** Let  $\varphi(x,y)$  be  $\Delta_0^b$ -formula. Assume  $S_2$  proves

$$\forall x \exists y \varphi(x,y)$$

What can you say about the deterministic/non-deterministic complexity of an algorithm computing y given x so that  $\varphi(x, y)$  holds?

**Exercise 4.** \* What about the deterministic space complexity of the above problems?

In particular, is it possible to find an efficiently decidable predicate P(x) which is impossible to express as a  $\Delta_0^b$ -formula?

**Definition 5.** We define the class  $\Sigma_1^b \subseteq \Delta_0$  as the smallest class of formulas satisfying conditions below

- $\Delta_0^b \subseteq \Sigma_1^b$
- for any  $\varphi, \psi \in \Sigma_1^b$ ,  $\varphi \wedge \psi \in \Sigma_1^b$  and  $\varphi \vee \psi \in \Sigma_1^b$
- for any  $\varphi(x) \in \Sigma_1^b$  and  $L_{S_2}$ -term t not containing  $x, \forall x \leq |t| \varphi(x) \in \Sigma_1^b$  and  $\exists x \leq |t| \varphi(x) \in \Sigma_1^b$
- for any  $\varphi(x) \in \Sigma_1^b$  and  $L_{S_2}$ -term t not containing  $x, \exists x \leq t \varphi(x) \in \Sigma_1^b$
- $\Sigma_1^b$  closed under logical equivalence

The class  $\Pi_1^b$  is defined similarly with fourth conditions replaced by

• for any  $\varphi(x) \in \Pi_1^b$  and  $L_{S_2}$ -term t not containing  $x, \forall x \leq t \varphi(x) \in \Pi_1^b$ 

**Exercise 6.** Show that  $\Pi_1^b$  contains exactly formulas which are negations of the formulas from  $\Sigma_1^b$ .

**Exercise 7.** Solve exercises 2 and 3 with  $\Delta_0^b$  replaced by  $\Sigma_1^b$  and  $\Pi_1^b$ .

Any efficiently decidable predicate (or efficiently computable function) can be expressed as a suitable  $\Sigma_1^b$ -formula in the language  $L_{S_2}$  which is a stronger statement than the one from the beginning.

However, any NP predicate or FNP relation is expressible as  $\Sigma_1^b$ , as well. What makes NP different from P, or FNP different from FP?

**Definition 8.** Formula  $\varphi(x) \in \Sigma_1^b$  is said to belong to  $\Delta_1^b$  if there is a formula  $\psi(x) \in \Pi_1^b$  so that

$$\forall x \varphi(x) \leftrightarrow \psi(x)$$

Formula  $\varphi(x,y) \in \Sigma_1^b$  is called total if

$$\forall x \exists y \phi(x,y)$$

Given a theory T in  $L_{S_2}$ , we say that a formula  $\varphi(x) \in \Sigma_1^b$  belongs to  $\Delta_1^b$  in T if

$$T \vdash \forall x \varphi(x) \leftrightarrow \psi(x)$$

and similarly  $\varphi(x,y) \in \Sigma_1^b$  is called total in T if

$$T \vdash \forall x \exists y \varphi(x, y)$$

Fact 9. Any efficiently decidable predicate can be expressed as a suitable  $\Delta_1^b$ -formula. Any efficiently computable function can be expressed as a suitable total  $\Sigma_1^b$ -formula.

**Theorem 10.** There is a theory  $S_2^1$  a subtheory of  $S_2$  for which the following holds

- a predicate P(x) is in P if and only if it is expressible as a formula  $\varphi(x)$  which is  $\Delta_1^b$  in  $S_2^1$
- a relation R(x,y) is in FP if and only if it is expressible as a formula  $\varphi(x,y) \in \Sigma_1^b$  which is total in  $S_2^1$

Exercise 11. Show that the above bullets are, in fact, equivalent. In other words, it is enough to focus only on total relations.