Sequent calculus LK

Propositional sequent calculus

Definition 1. Let $A_1, \dots, A_n$ and $B_1, \dots, B_m$ be propositional formulas. A sequent is a symbol of the form

$$A_1, \dots, A_n \longrightarrow B_1, \dots, B_m.$$

The semantics for a sequent are the same as for the formula

$$\bigwedge_i A_i \to \bigvee_i B_i,$$

which is semantically equivalent to

$$\bigvee_i \neg A_i \vee \bigvee_i B_i.$$

Definition 2. The Sequent calculus is a propositional proof system (which proves sequents), whose proves are given as follows.

A proof of a sequent S is a sequence of sequents, $S_1, \dots, S_k$, where $S_k = S$ and each S_i is either an *initial sequent*

$$x \longrightarrow x,$$

where x is a propositional variable or was derived from $S_j, S_l, 1 \leq j \leq l \leq$ by one of the following rules.

Weak Structural Rules

$$(\text{Exchange:L})\frac{\Gamma, A, B, \Pi \longrightarrow \Delta}{\Gamma, B, A, \Pi \longrightarrow \Delta} \qquad (\text{Exchange:R})\frac{\Gamma \longrightarrow \Delta, A, B, \Lambda}{\Gamma \longrightarrow \Delta, B, A, \Lambda}$$

$$(\text{Contraction:L})\frac{\Gamma, A, A, \Pi \longrightarrow \Delta}{\Gamma, A, \Pi \longrightarrow \Delta} \qquad (\text{Contraction:R})\frac{\Gamma \longrightarrow \Delta, A, A, \Lambda}{\Gamma \longrightarrow \Delta, A, \Lambda}$$

$$(\text{Weakening:L})\frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta} \qquad (\text{Weakening:R})\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, A}$$

The Cut Rule

$$(\text{Cut})\frac{\Gamma \longrightarrow \Delta, A \qquad \Gamma, A \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}$$

The Propositional Rules

$$(\neg\text{:L})\frac{\Gamma \longrightarrow \Delta, A}{\Gamma, \neg A \longrightarrow \Delta} \qquad (\neg\text{:R})\frac{\Gamma, A \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A}$$

$$(\wedge\text{:L})\frac{\Gamma, A, B \longrightarrow \Delta}{\Gamma, A \wedge B \longrightarrow \Delta} \qquad (\wedge\text{:R})\frac{\Gamma \longrightarrow \Delta, A \qquad \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, A \wedge B}$$

$$(\vee\text{:L})\frac{\Gamma, A \longrightarrow \Delta \qquad \Gamma, B \longrightarrow \Delta}{\Gamma, A \vee B \longrightarrow \Delta} \qquad (\vee\text{:R})\frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, A \vee B}$$

Sequent calculus is denoted LK (for Logischer Kalkülus) or PK for the propositional version.

Fact 3. $LK \equiv_p F$

Definition 4. LK^- is the subsystem of LK, which forbids the use of the cut rule.

Exercise 5. Prove $LK^- \vdash \longrightarrow A \vee \neg A$

Exercise 6. Prove $LK^- \vdash \longrightarrow (A \vee \neg A) \wedge (B \vee \neg B)$

Exercise 7. Prove $LK^- \vdash \longrightarrow (A \wedge B) \vee (A \wedge \neg B) \vee (A \wedge \neg B) \vee (\neg A \wedge \neg B)$

Exercise 8. Prove LK^- is complete.

First order sequent calculus

Definition 9. Let L be a first order language. The first order sequent calculus, also labeled LK, is a first order proof system, which is an extension of the propositional LK where we replace the propositional variables by atomic L-formulas.

The additional rules are **Quantifier Rules**

$$(\forall\text{:L})\frac{\Gamma, A(t) \longrightarrow \Delta}{\Gamma, (\forall x)A(x) \longrightarrow \Delta} \qquad (\forall\text{:R})\frac{\Gamma \longrightarrow \Delta, A(b)}{\Gamma \longrightarrow \Delta, (\forall x)A(x)}$$

$$(\exists\text{:L})\frac{\Gamma, A(b) \longrightarrow \Delta}{\Gamma, (\exists x)A(x) \longrightarrow \Delta} \qquad (\exists\text{:R})\frac{\Gamma \longrightarrow \Delta, A(b)}{\Gamma \longrightarrow \Delta, (\exists x)A(x)},$$

where Γ and Δ are sequences of formulas, t is an arbitrary term, b is a free variable which does not occur in neither Γ nor Δ, b is called the *eigenvariable* of the rule.

Note that LK has no built-in axioms for equality, they have to be added as a part of the studied theory.

Exercise 10. Prove that if $LK \vdash \Gamma \longrightarrow \Delta$ without the cut rule, then every sequent in this proofs contains only formulas which are subformulas of the formulas in Γ or Δ.

Theorem 11. (Cut-free completeness) If a theory T semantically implies

$$\Gamma \longrightarrow \Delta,$$

then there are $A_1, \ldots, A_n \in T$ such that there is a cut free proof of

$$A_1, \ldots, A_n, \Gamma \longrightarrow \Delta.$$

Exercise 12. If the axioms of groups (including the axioms of equality) prove $\longrightarrow t = s$, where t and s are some group terms, then there is an LK-proof of this equality, where every formula in every sequent is